ON THE FREE MODULAR LATTICE GENERATED BY 2+1+1

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In this paper, we shall solve the probrem 29 in Lattice Theory by G. Birkhoff.

DEFINITION 1. A lattice is a system of elements which satisfy the following identities.

L_1. $x \cup x = x$, and $x \cap x = x$.

L_2. $x \cup y = y \cup x$, and $x \cap y = y \cap x$.

L_3. $x \cup (y \cup z) = (x \cup y) \cup z$, and $x \cap (y \cap z) = (x \cap y) \cap z$.

L_4. $x \cup (x \cap y) = x$, and $x \cap (x \cup y) = x$.

DEFINITION 2. A lattice is called modular if and only if its elements satisfy the following modular identity.

L_5. If $x \leqq z$, then $x \cup (y \cap z) = (x \cup y) \cap z$.

THEOREM 1*. The free modular lattice generated by $x_1 > x_2 > x_3$ and $y_1 > y_2$ has 33 elements.

THEOREM 2**. The free modular lattice generated by $x_1 > x_2 > x_3$ and $y_1 > y_2 > y_3$ has 68 elements.

We shall now determin the free modular lattice generated by 2+1+1.

THEOREM 3. The free modular lattice generated by $x_1 > x_2$, α and β has 138 elements, and the diagram of the following figure.

Proof. Let us agree to the following designation.

$I = x_1 \cup \alpha \cup \beta$,

$a_1 = x_1 \cup \alpha$,

$a_2 = x_1 \cup \beta,$

$a_3 = x_2 \cup \alpha \cup \beta,$

$b_1 = x_1 \cup ((x_1 \cup \alpha) \cap \beta),$

$b_2 = x_2 \cup (x_1 \cap (\alpha \cup \beta)) \cup \alpha,$

$b_3 = x_2 \cup (x_1 \cap (\alpha \cup \beta)) \cup \beta,$

$b_4 = \alpha \cup \beta,$

$c_1 = x_1 \cup ((x_2 \cup \alpha) \cap \beta) \cup ((x_2 \cup \beta) \cap \alpha),$

$c_2 = x_2 \cup (x_1 \cap \beta) \cup \alpha,$

$c_3 = x_2 \cup (x_1 \cap (\alpha \cup \beta)) \cup ((x_1 \cup \alpha) \cap \beta),$

$c_4 = (x_1 \cap (\alpha \cup \beta)) \cup \alpha,$

$c_5 = x_2 \cup (x_1 \cap \alpha) \cup \beta,$

$c_6 = (x_1 \cap (\alpha \cup \beta)) \cup \beta,$

$d_1 = x_1 \cup (\alpha \cap \beta),$

$d_2 = x_2 \cup (x_1 \cap (\alpha \cup \beta)) \cup ((x_2 \cup \beta) \cap \alpha) \cup ((x_2 \cup \alpha) \cap \beta),$

$d_3 = x_2 \cup \alpha,$

$d_4 = x_2 \cup (x_1 \cap \beta) \cup ((x_1 \cup \beta) \cap \alpha),$

$d_5 = (x_2 \cap (\alpha \cup \beta)) \cup (x_1 \cap \beta) \cup \alpha,$

$d_6 = x_2 \cup (x_1 \cap \alpha) \cup ((x_1 \cup \alpha) \cap \beta),$

$d_7 = (x_1 \cap (\alpha \cup \beta) \cup ((x_1 \cup \alpha) \cap \beta),$

$d_8 = x_2 \cup \beta,$

$d_9 = (x_2 \cap (\alpha \cup \beta)) \cup (x_1 \cap \alpha) \cup \beta,$

$e_1 = x_2 \cup (x_1 \cap (\alpha \cup \beta)) \cup (\alpha \cap \beta),$

$e_2 = x_2 \cup ((x_1 \cup \beta) \cap \alpha),$

$e_3 = (x_2 \cap (\alpha \cup \beta)) \cup \alpha,$

$e_4 = x_2 \cup (x_1 \cap \alpha) \cup (x_1 \cap \beta) \cup ((x_2 \cup \alpha) \cap \beta) \cup ((x_2 \cup \beta) \cap \alpha),$

$e_5 = (x_1 \cap (\alpha \cup \beta)) \cup ((x_2 \cup \beta) \cap \alpha) \cup ((x_2 \cup \alpha) \cap \beta),$

$e_6 = (x_2 \cap (\alpha \cup \beta)) \cup (x_1 \cap \beta) \cup ((x_1 \cup \beta) \cap \alpha),$

$e_7 = (x_1 \cap \beta) \cup \alpha,$

$e_8 = x_2 \cup ((x_1 \cup \alpha) \cap \beta),$

$e_9 = (x_2 \cap (\alpha \cup \beta)) \cup (x_1 \cap \alpha) \cup ((x_1 \cup \alpha) \cap \beta),$

$e_{10} = (x_2 \cap (\alpha \cup \beta)) \cup \beta.$

$e_{11} = (x_1 \cap \alpha) \cup \beta,$

$f_1 = x_2 \cup (x_1 \cap (\alpha \cup \beta)),$

$f_2 = x_2 \cup (x_1 \cap \alpha) \cup (x_1 \cap \beta) \cup (\alpha \cap \beta),$

$f_3=(x_1\cap(\alpha\cup\beta))\cup(\alpha\cap\beta).$

$f_4=x_2\cup(x_1\cap\alpha)\cup((x_2\cup\alpha)\cap\beta),$

$f_5=(x_2\cap(\alpha\cup\beta))\cup((x_1\cup\beta)\cap\alpha),$

$f_6=(x_1\cap\beta\cap(x_2\cup\alpha))\cup\alpha,$

$f_7=x_2\cup(x_1\cap\beta)\cup((x_2\cup\beta)\cap\alpha),$

$f_8=(x_1\cap\alpha)\cup(x_1\cap\beta)\cup((x_2\cup\alpha)\cap\beta)\cup((x_2\cup\beta)\cap\alpha),$

$f_9=(x_1\cap\beta)\cup((x_1\cup\beta)\cap\alpha),$

$f_{10}=(x_2\cap(\alpha\cup\beta))\cup((x_1\cup\alpha)\cap\beta),$

$f_{11}=(x_1\cap\alpha)\cup((x_1\cup\alpha)\cap\beta),$

$f_{12}=(x_1\cap\alpha\cap(x_2\cup\beta))\cup\beta,$

$g_1=x_2\cup(x_1\cap\alpha)\cup(x_1\cap\beta),$

$g_2=x_2\cup(x_1\cap\alpha)\cup(\alpha\cap\beta),$

$g_3=x_2\cup(x_1\cap\beta)\cup(\alpha\cap\beta),$

$g_4=(x_2\cap(\alpha\cup\beta))\cup(x_1\cap\alpha)\cup(x_1\cap\beta)\cup(\alpha\cap\beta),$

$g_5=x_2\cup((x_2\cup\alpha)\cap\beta)\cup((x_2\cup\beta)\cap\alpha),$

$g_6=(x_1\cap\alpha)\cup((x_2\cup\alpha)\cap\beta)\cup((x_2\cup\beta)\cap\alpha),$

$g_7=(x_1\cap\beta\cap(x_2\cup\alpha))\cup((x_1\cup\beta)\cap\alpha),$

$g_8=(x_2\cap\beta)\cup\alpha,$

$g_9=(x_1\cap\beta)\cap((x_2\cup\alpha)\cap\beta)\cup((x_2\cup\beta)\cap\alpha),$

$g_{10}=(x_1\cap\alpha)\cup(x_1\cap\beta)\cup((x_2\cup\beta)\cap\alpha),$

$g_{11}=(x_1\cap\alpha)\cup(x_1\cap\beta)\cup((x_2\cup\alpha)\cap\beta),$

$g_{12}=(x_1\cap\alpha\cap(x_2\cup\beta))\cup((x_1\cup\alpha)\cap\beta),$

$g_{13}=(x_2\cap\alpha)\cup\beta,$

$h_1=x_2\cup(x_1\cap\beta\cap(x_2\cup\alpha))\cup(\alpha\cap\beta),$

$h_2=((x_2\cup\alpha)\cap\beta)\cup((x_2\cup\beta)\cap\alpha),$

$l=x_2\cup(\alpha\cap\beta),$

$m_1=x_2\cup(x_1\cap\alpha),$

$m_2=x_2\cup(x_1\cap\beta),$

$m_3=(x_2\cap(\alpha\cup\beta))\cup(x_1\cap\alpha)\cup(\alpha\cap\beta),$

$m_4=(x_2\cap(\alpha\cup\beta))\cup(x_1\cap\beta)\cup(\alpha\cap\beta),$

$m_5=(x_1\cap\alpha)\cup(x_1\cap\beta\cap(x_2\cup\alpha))\cup((x_2\cup\beta)\cap\alpha),$

$m_6=(x_1\cap\alpha)\cup((x_2\cup\alpha)\cap\beta),$

$m_7=(x_2\cap\beta)\cup((x_1\cup\beta)\cap\alpha),$

$m_8=(x_1\cap\beta)\cup((x_2\cup\beta)\cap\alpha),$

$m_9 = (x_1 \cap \beta) \cup ((x_1 \cap \alpha \cap (x_2 \cup \beta)) \cup ((x_2 \cup \alpha) \cap \beta)$,

$m_{10} = (x_2 \cap \alpha) \cup ((x_1 \cup \alpha) \cap \beta$,

$O = x_2 \cap \alpha \cap \beta$,

$a'_1 = x_2 \cap \alpha$,

$a'_2 = x_2 \cap \beta$,

$a'_3 = x_1 \cap \alpha \cap \beta$,

$b'_1 = x_2 \cap ((x_2 \cap \alpha) \cup \beta)$,

$b'_2 = x_1 \cap (x_2 \cup (\alpha \cap \beta)) \cap \alpha$,

$b'_3 = x_1 \cap (x_2 \cup (\alpha \cap \beta)) \cap \beta$,

$b'_4 = \alpha \cap \beta$,

$c'_1 = x_2 \cap ((x_1 \cap \alpha) \cup \beta) \cap ((x_1 \cap \beta) \cup \alpha)$,

$c'_2 = x_1 \cap (x_2 \cup \beta) \cap \alpha$,

$c'_3 = (x_2 \cup (\alpha \cap \beta)) \cap ((x_2 \cap \alpha) \cup \beta) \cap x_1$,

$c'_4 = (x_2 \cup (\alpha \cap \beta)) \cap \alpha$,

$c'_5 = x_1 \cap (x_2 \cup \alpha) \cap \beta$,

$c'_6 = (x_2 \cup (\alpha \cap \beta)) \cap \beta$,

$d'_1 = x_2 \cap (\alpha \cup \beta)$,

$d'_2 = x_1 \cap (x_2 \cup (\alpha \cap \beta)) \cap ((x_1 \cap \beta) \cup \alpha) \cap ((x_1 \cap \alpha) \cup \beta)$,

$d'_3 = x_1 \cap \alpha$,

$d'_4 = x_1 \cap (x_2 \cup \beta) \cap ((x_2 \cap \beta) \cup \alpha)$,

$d'_5 = (x_1 \cup (\alpha \cap \beta)) \cap (x_2 \cup \beta) \cap \alpha$,

$d'_6 = x_1 \cap (x_2 \cup \alpha) \cap ((x_2 \cap \alpha) \cup \beta)$,

$d'_7 = (x_2 \cup (\alpha \cap \beta)) \cap ((x_2 \cap \alpha) \cup \beta)$,

$d'_8 = x_1 \cap \beta$,

$d'_9 = (x_1 \cup (\alpha \cap \beta)) \cap (x_2 \cup \alpha) \cap \beta$,

$e'_1 = x_1 \cap ((x_2 \cup (\alpha \cap \beta)) \cap (\alpha \cup \beta)$,

$e'_2 = x_1 \cap ((x_2 \cap \beta) \cup \alpha)$,

$e'_3 = (x_1 \cup (\alpha \cap \beta)) \cap \alpha$,

$e'_4 = x_1 \cap (x_2 \cup \alpha) \cap (x_2 \cup \beta) \cap ((x_1 \cap \alpha) \cup \beta) \cap ((x_1 \cap \beta) \cup \alpha)$,

$e'_5 = (x_2 \cup (\alpha \cap \beta)) \cap ((x_1 \cap \beta) \cup \alpha) \cap ((x_1 \cap \alpha) \cup \beta)$,

$e'_6 = (x_1 \cup (\alpha \cap \beta)) \cap (x_2 \cup \beta) \cap ((x_2 \cap \beta) \cup \alpha)$,

$e'_7 = (x_2 \cup \beta) \cap \alpha$,

$e'_8 = x_1 \cap ((x_2 \cap \alpha) \cup \beta)$,

$e'_9 = (x_1 \cup (\alpha \cap \beta)) \cap (x_2 \cup \alpha) \cap ((x_2 \cap \alpha) \cup \beta)$,

$e'_{10}=(x_1\cup(\alpha\cap\beta))\cap\beta,$

$e'_{11}=(x_2\cup\alpha)\cap\beta,$

$f'_1=x_1\cap(x_2\cup(\alpha\cap\beta)),$

$f'_2=x_1\cap(x_2\cup\alpha)\cap(x_2\cup\beta)\cap(\alpha\cup\beta),$

$f'_3=(x_2\cup(\alpha\cap\beta))\cap(\alpha\cup\beta),$

$f'_4=x_1\cap(x_2\cup\alpha)\cap((x_1\cap\alpha)\cup\beta),$

$f'_5=(x_1\cup(\alpha\cap\beta))\cap((x_2\cap\beta)\cup\alpha),$

$f'_6=(x_2\cup\beta\cup(x_1\cap\alpha))\cap\alpha,$

$f'_7=x_1\cap(x_2\cup\beta)\cap((x_1\cap\beta)\cup\alpha),$

$f'_8=(x_2\cup\alpha)\cap(x_2\cup\beta)\cap((x_1\cap\alpha)\cup\beta)\cap((x_1\cap\beta)\cup\alpha),$

$f'_9=(x_2\cup\beta)\cap((x_2\cap\beta)\cup\alpha),$

$f'_{10}=(x_1\cup(\alpha\cap\beta))\cap((x_2\cap\alpha)\cup\beta),$

$f'_{11}=(x_2\cup\alpha)\cap((x_2\cap\alpha)\cup\beta),$

$f'_{12}=(x_2\cup\alpha\cup(x_1\cap\beta))\cap\beta,$

$g'_1=x_1\cap(x_2\cup\alpha)\cap(x_2\cup\beta),$

$g'_2=x_1\cap(x_2\cup\alpha)\cap(\alpha\cup\beta),$

$g'_3=x_1\cap(x_2\cup\beta)\cap(\alpha\cup\beta),$

$g'_4=(x_1\cup(\alpha\cap\beta))\cap(x_2\cup\alpha)\cap(x_2\cup\beta)\cap(\alpha\cup\beta),$

$g'_5=x_1\cap((x_1\cap\alpha)\cup\beta)\cap((x_1\cap\beta)\cup\alpha),$

$g'_6=(x_2\cup\alpha)\cap((x_1\cap\alpha)\cup\beta)\cap((x_1\cap\beta)\cup\alpha),$

$g'_7=(x_2\cup\beta\cup(x_1\cap\alpha))\cap((x_2\cap\beta)\cup\alpha),$

$g'_8=(x_1\cup\beta)\cap\alpha,$

$g'_9=(x_2\cup\beta)\cap((x_1\cap\alpha)\cup\beta)\cap((x_1\cap\beta)\cup\alpha),$

$g'_{10}=(x_2\cup\beta)\cap(x_2\cup\alpha)\cap((x_1\cap\beta)\cup\alpha),$

$g'_{11}=(x_2\cup\alpha)\cap(x_2\cup\beta)\cap((x_1\cap\alpha)\cup\beta),$

$g'_{12}=(x_2\cup\alpha\cup(x_1\cap\beta))\cap((x_2\cap\alpha)\cup\beta),$

$g'_{13}=(x_1\cup\alpha)\cap\beta,$

$h'_1=x_1\cap(x_2\cup\beta\cup(x_1\cap\alpha))\cap(\alpha\cup\beta),$

$h'_2=((x_1\cap\alpha)\cup\beta)\cap((x_1\cap\beta)\cup\alpha),$

$l'=x_1\cap(\alpha\cup\beta).$

It is obviously that the join and meet-functions diagramed in figure are all consequence of $L_1\sim L_5$.

In fact,

$b_1=x_1\cup g'_{13}=a_1\cap a_2,$

$b_2=x_2\cup\alpha\cup l'=a_1\cap a_3,$

$b_3=x_2\cup\beta\cup l'=a_2\cap a_3,$

$c_3=x_2\cup l'\cup g'_{13}=a_1\cap a_2\cap a_3,$

$c_4=l'\cup\alpha=a_1\cap b_4,$

$c_6=l'\cup\beta=a_2\cap b_4,$

$d_2=x_2\cup l'\cup e'_7\cup e'_{11}=a_3\cap c_1,$

$d_4=x_2\cup d'_3\cup g'_8=a_2\cap c_2,$

$d_5=d'_1\cup d'_8\cup\alpha=b_4\cap c_2,$

$d_6=x_2\cup d'_3\cup g'_{13}=a_1\cap c_5,$

$d_7=l'\cup g'_{13}=b_1\cap b_4,$

$d_9=d'_1\cup d'_3\cup\beta=c_5\cap b_4,$

$e_1=x_2\cup l'\cup b'_4=d_1\cap a_3,$

$e_2=x_2\cup g'_8=a_2\cap d_3,$

$e_3=d'_1\cup\alpha=d_3\cap b_4,$

$e_4=x_2\cup d'_3\cup d'_8\cup e'_{11}\cup e'_7=c_5\cap c_2,$

$e_5=l'\cup e'_{11}\cup e'_7=c_1\cap b_4,$

$e_6=d'_1\cup d'_8\cup g'_8=d_4\cap b_4,$

$e_8=x_2\cup g_{13}'=a_1\cap d_8,$

$e_9=d_1'\cup d_3'\cup g_{13}'=d_6\cap b_4,$

$e_{10}=d'_1\cup\beta=d_3\cap b_4,$

$f_1=x_2\cup l'=x_1\cap a_3,$

$f_2=x_2\cup d'_3\cup d'_8\cup b'_4=d_1\cap c_2,$

$f_3=l'\cup b'_4=d_1\cap b_4,$

$f_4=x_2\cup d'_3\cup e'_{11}=d_3\cap c_5,$

$f_5=d'_1\cup g'_8=e_2\cap b_4,$

$f_6=c'_5\cup\alpha=d_3\cap e_7,$

$f_7=x_2\cup d'_8\cup e'_7=c_2\cap d_3,$

$f_8=d'_3\cup d'_8\cup e'_{11}\cup e'_7=e_4\cap b_4,$

$f_9=d'_8\cup g'_8=a_2\cap e_7,$

$f_{10}=d'_1\cup g'_{13}=e_3\cap b_4,$

$f_{11}=d'_3\cup g'_{13}=a_1\cap e_{11},$

$f_{12}=c'_2\cup\beta=d_3\cap e_{11}$

$g_1=x_2\cup d'_3\cup d'_8=x_1\cap c_2,$

$b'_1=x_2\cap g_{13}=a'_1\cup a'_2,$

$b'_2=x_1\cap\alpha\cap l=a'_1\cup a'_3,$

$b'_3=x_1\cap\beta\cap l=a'_2\cup a'_3,$

$c'_3=x_1\cap l\cap g_{13}=a'_1\cup a'_2\cup a'_3,$

$c'_4=l\cap\alpha=a'_1\cup b'_4,$

$c'_6=l\cap\beta=a'_2\cup b'_4,$

$d'_2=x_1\cap l\cap e_7\cap e_{11}=a'_3\cup c'_1,$

$d'_4=x_1\cap d_3\cap g_8=a'_2\cup c'_2,$

$d'_5=d_1\cap d_8\cap\alpha=b'_4\cup c'_2,$

$d'_6=x_1\cap d_3\cap g_{13}=a'_1\cup c'_5,$

$d'_7=l\cap g_{13}=b'_1\cup b'_4,$

$d'_9=d_1\cap d_3\cap\beta=c'_5\cup b'_4,$

$e'_1=x_1\cap l\cap b_4=d'_1\cup a'_3,$

$e'_2=x_1\cup g_8=a'_2\cup d'_3,$

$e'_3=d_1\cap\alpha=d'_3\cup b'_4,$

$e'_4=x_1\cap d_3\cap d_8\cap e_{11}\cap e_7=c_5\cup c_2,$

$e'_5=l\cap e_{11}\cap e_7=c'_1\cup b'_4,$

$e'_6=d_1\cap d_8\cap g_8=d'_4\cup b'_4,$

$e'_8=x_1\cap g_{13}=a'_1\cup d'_8,$

$e'_9=d_1\cap d_3\cap g_{13}=d'_6\cup b'_4,$

$e'_{10}=d_1\cap\beta=d'_8\cup b'_4,$

$f'_1=x_1\cap l=x'_2\cup a'_3,$

$f'_2=x_1\cap d_3\cap d_8\cap b_4=d'_2\cup c'_2,$

$f'_3=l\cap b_4=d'_1\cup b'_4,$

$f'_4=x_1\cap d_3\cap e_{11}=d'_3\cup c'_5,$

$f'_5=d_1\cap g_8=e'_2\cup b'_4,$

$f'_6=c_5\cap\alpha=d'_3\cup e'_7,$

$f'_7=x_1\cap d_8\cap e_7=c'_2\cup d'_8,$

$f'_8=d_3\cap d_8\cap e_{11}\cap e_7=e'_4\cup b'_4,$

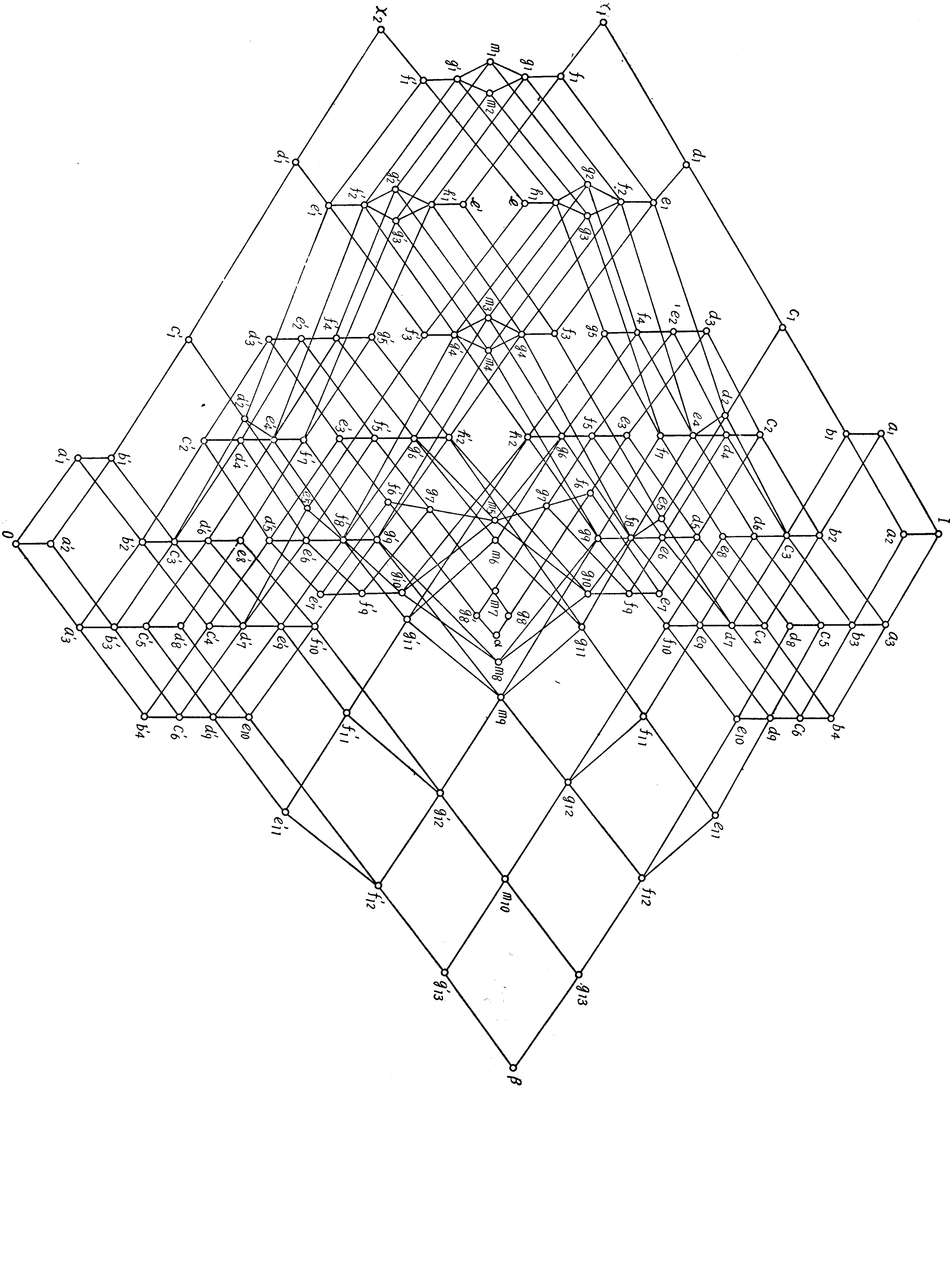
$f'_9=d_8\cap g_8=a'_2\cup e'_7,$

$f'_{10}=d_1\cap g_{13}=e'_8\cup b'_4,$

$f'_{11}=d_3\cap g_{13}=a'_1\cup e'_{11},$

$f'_{12}=c_2\cap\beta=d'_8\cup e'_{11},$

$g'_1=x_1\cap d_3\cap d_8=x_2\cup c'_2,$



$g_2=x_2\cup d'_3\cup b'_4=d_1\cap d_3,$

$g_3=x_2\cup d'_8\cup b'_4=d_1\cap d_8,$

$g_4=d'_1\cup d'_3\cup d'_8\cup b'_4=f_2\cap b_4,$

$g_5=x_2\cup e'_{11}\cup e'_7=d_3\cap d_8,$

$g_6=d'_3\cup e'_{11}\cup e'_7=d_3\cap c_5\cap b_4,$

$g_7=c'_5\cup g'_8=d_3\cap a_2\cap e_7,$

$g_9=d'_8\cup e'_7\cup e'_{11}=d_8\cap c_2\cup b_4,$

$g_{10}=d'_2\cup d'_8\cup e'_7=c_5\cap e_7,$

$g_{11}=d'_3\cup d'_8\cup e'_{11}=c_2\cap e_{11},$

$g_{12}=c'_2\cup g_{13}=d_8\cap a_1\cap e_{11},$

$h_1=x_2\cup c'_5\cup b'_4=d_1\cap d_3\cap d_8,$

$h_2=e'_7\cup e'_{11}=d_3\cap d_8\cap b_4,$

$g'_2=x_1\cap d_3\cap b_4=d'_1\cup d'_3,$

$g'_3=x_1\cap d_8\cap b_4=d'_1\cup d'_8,$

$g'_4=d_1\cap d_3\cap d_8\cap b_4=f'_2\cup b'_4,$

$g'_5=x_1\cap e_{11}\cap e_7=d'_3\cup d'_8,$

$g'_6=d_3\cap e_{11}\cap e_7=d'_3\cup c'_5\cup b'_4,$

$g'_7=c_5\cap g_8=d'_3\cup a'_2\cup e'_7,$

$g'_9=d_8\cap e_7\cap e_{11}=d'_8\cup c'_2\cup b'_4,$

$g'_{10}=d_3\cap d_8\cup e_7=c'_5\cup e'_7,$

$g'_{11}=d_3\cap d_8\cap e_{11}=c'_2\cup e'_{11},$

$g'_{12}=c_2\cap g_{13}=d'_8\cup a'_1\cup e'_{11},$

$h'_1=x_1\cap c_5\cap b_4=d'_1\cup d'_3\cup d'_8,$

$h'_2=e_7\cap e_{11}=d'_3\cup d'_8\cup b'_4,$

$m_1=x_2\cup d'_3=x_1\cap d_3,$

$m_2=x_2\cup d'_8=x_1\cap d_8,$

$m_3=d'_1\cup d'_3\cup b'_4=d_1\cap d_3\cap b_4,$

$m_5=d'_3\cup c'_5\cup e'_7=d_3\cap c_5\cap e_7,$

$m_6=d'_3\cup e'_{11}=d_3\cap e_{11},$

$m_7=a'_2\cup g'_8=a_2\cap g_8,$

$m_8=d'_8\cap e'_7=d_8\cap e_7,$

$m_9=d'_8\cup c'_2\cup e'_{11}=d_8\cap c_2\cap e_{11},$

$m_{10}=a'_1\cup g'_{13}=a_1\cap g_{13}.$

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*, **. These results are due to author: cf. Annual Reports of Studies, Osaka Joshigakuen Junior College, No. 2, (1958).